

On the Interpretation of the Lattice of Subspaces of the Hilbert Space as a Propositional Calculus

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In the lattice of subspaces of the Hilbert space elements can be defined which may be considered as generalized implications. It is shown, that these elements satisfy the most important relations which are known to be valid for the classical implication. These results seem to justify the interpretation of this lattice as a propositional calculus sometime called quantum logic.

Introduction

G. BIRKHOFF and J. v. NEUMANN¹ have shown in 1936 that the closed linear manifolds (subspaces) of the Hilbert space form a complete atomic and ortho-complemented lattice L_0 which at least for finite dimensional subspaces is also modular. JAUCH^{2,3}, PIRON⁴ and KAMBER⁵ have pointed out, that instead of the modularity a weaker condition can be formulated in L_0 which is always fulfilled for the lattice of subspaces of the Hilbert space, and which has been called *weak modularity*² or *quasi-modularity*⁵. A lattice L_0 which is quasimodular will be denoted here by L_q .

Many kinds of lattices, for instance the Boolean lattices, can be interpreted as logical algebras, often called propositional calculi. The question whether the lattice of subspaces of the Hilbert space can also be interpreted as a propositional calculus has been raised already by BIRKHOFF and v. NEUMANN¹. In recent years an attempt has been made to justify such an interpretation in the framework of the operational foundation⁶ of logic⁷. From the point of view of lattice theory the possibility of a logical interpretation of a lattice depends essentially on the question, whether in the lattice considered elements can be defined which have the most important properties of the *implication*^{8,9} in ordinary logic. (§1) This has been pointed out by KUNSEMÜLLER¹⁰ and more recently by JAUCH and PIRON¹¹. As a first result in this respect it could be shown¹², that in a quasimodular lattice L_q it is always possible for two elements a and b to define an other element $q(a, b)$ which is equal to the unit if and only if $a \leq b$,

where " \leq " means the partly ordering relation in the lattice.

In ordinary logic it is well known which properties a lattice must have to allow for the definition of elements which satisfy all the relations which are valid for the implication in a propositional calculus⁸. These lattices are called *implicative* or *relatively pseudo-complemented*⁹. Since an implicative lattice can be shown to be distributive it is quite clear that in the lattice L_q , which is not distributive, it is not possible to define an implication strictly in the sense of ordinary logic. However, the results mentioned above¹² motivate the search for a convenient generalization of the implication which exists also in the lattice of subspaces of the Hilbert space.

In this paper we investigate the problem whether in the lattice L_q an element can be defined which has the essential properties of the implication. Starting from L_0 we show that two postulates can be formulated which guarantee the existence of a lattice element q with the most important properties of the implication and which are also valid in L_q . Moreover it will be shown that these postulates are necessary and sufficient for the quasimodularity of the lattice (theorem I and II) and that the element q which we call *quasi-implication* is uniquely defined by the postulates (theorem III). Furthermore a number of interesting relations which are known to be valid for the implication in ordinary logic can also be proved for the quasi-implication (theorem IV, V, VI). These results seem to justify the attempt to interpret the lattice of subspaces of the Hilbert space as a propositional calculus, sometimes called *quantum logic*.

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§ 1. The Relation between Lattice Theory and Logic

A propositional calculus can be considered as a lattice L the elements of which are the propositions a, b, c . In respect to the relation $a \leq b$ (a implies b) the propositions form a partly ordered system and in respect to the operations $a \wedge b$ (a and b) and $a \vee b$ (a or b) a lattice, since according to the logical rules the element $a \wedge b$ is the greatest lower bound and the element $a \vee b$ is the least upper bound of the elements a and b . Further axioms in the propositional calculus lead to more complicated lattices. For instance the intuitionistic propositional calculus corresponds to an implicative lattice whereas the classical propositional calculus corresponds to a Boolean lattice.

The inverse relation is much more complicated. The reason is that in all known logical systems the implication $a \leq b$ is also used in iterated form, for instance in the well known *modus ponens* law $(a \wedge (a \leq b)) \leq b$. In fact, it is difficult to speak of a logical calculus without the modus ponens, since this rule is essential, if one wants to deduce a proposition from other propositions. Therefore a lattice can only be interpreted as a propositional calculus if at least the modus ponens law can be expressed in terms of lattice theory. For the interpretation of the modus ponens in a propositional calculus it is important that one may consider the relation $a \leq b$ also as a proposition — sometimes called material implication⁸ or conditional¹¹ — which is true if and only if the relation $a \leq b$ holds. Consequently a lattice which can be considered as a propositional calculus must have the property that for any two elements a and b there exists another element $c(a, b)$ which satisfies the relation $a \wedge c \leq b$ and which is equal to the unit if and only if the relation $a \leq b$ is valid.

Lattices which have this property have been investigated in detail and are known as *implicative*⁸ or *relatively pseudo-complemented* lattices⁹. A lattice L in which for any pair a, b of elements there exists an element $c(a, b)$ which satisfies the two relations

$$a \wedge c(a, b) \leq b, \quad (1.1)$$

$$a \wedge x \leq b \Rightarrow x \leq c(a, b) \quad (1.2)$$

is called an implicative lattice L_I and the element $c(a, b)$ implication. If we adopt the postulates (1.1) and (1.2) it can be shown that

- a) the lattice is distributive,
- b) in L_I there is only one element $c(a, b)$ which satisfies (1.1) and (1.2),
- c) $a \leq b$ if and only if $\mathbf{1} = c(a, b)$, where $\mathbf{1} = c(a, a)$ is the unit element which always exists in a lattice L_I .

If in addition to the axioms (1.1) and (1.2) we adopt the postulate (Peirce's law)

$$c(c(a, b), a) \leq a \quad (1.3)$$

the lattice is called a *classical implicative* lattice. It can be shown, that in L_I (1.3) is equivalent to

$$a \vee c(a, b) = \mathbf{1}. \quad (1.4)$$

Furthermore it follows that a classical implicative lattice with zero element $\mathbf{0}$ is a Boolean lattice, if the complement a' of the element a is defined by $a' = c(a, \mathbf{0})$. In such a lattice the implication is given by

$$c(a, b) = a' \vee b \quad (1.5)$$

and from (1.4) the *tertium non datur* $a' \vee a = \mathbf{1}$ can be obtained.

§ 2. The Lattice of Subspaces of the Hilbert Space

The closed linear manifolds (subspaces) of the Hilbert space H which will be denoted here by a, b, c, \dots form a partly ordered system in respect to the relation R given by $a \leq b$. The intersection $a \wedge b$ of the subspaces a and b is — in respect to R — the greatest lower bound of a and b and the subspace $a \vee b$ spanned by a and b is the least upper bound of the two elements. Therefore the subspaces form a lattice with respect to the relation \leq and the operations \wedge and \vee . The zero element $\mathbf{0}$ is given by the empty set and the unit $\mathbf{1}$ by the Hilbert space H .

The lattice of the subspaces of the Hilbert space is σ -complete i.e. for any countable set of subspaces of L there is a least upper bound and a greatest lower bound. Furthermore L is atomic, i.e. for any $x \in L$, there is an atom $a \leq x$ with the property, that from $\mathbf{0} \leq z \leq a$ it follows $z = a$ or $z = \mathbf{0}$. For the following discussion it is very important that the lattice L is orthocomplemented, i.e. there is a mapping $a \rightarrow a' \in L$ such that

$$\begin{aligned} (a')' &= a, \\ a \wedge a' &= \mathbf{0}, \\ a \leq b &\Leftrightarrow b' \leq a'. \end{aligned} \quad (2.1)$$

From these relations it follows that

$$\begin{aligned}(a \wedge b)' &= a' \vee b', \\ (a \vee b)' &= a' \wedge b'\end{aligned}\quad (2.2)$$

and on account of $\theta' = \mathbf{1}$

$$a \vee a' = \mathbf{1}. \quad (2.3)$$

If a is a subspace of H the orthocomplement a' is given by the subspace which is completely orthogonal to a .

In addition to (2.1) the lattice L is quasimodular⁵, i.e.

$$\alpha \leq \beta, \gamma \leq \beta' \Rightarrow \beta \wedge (\alpha \vee \gamma) = \alpha. \quad (2.4)$$

This property (2.4) has also been called weak modularity², since in any modular lattice the relation (2.4) is always satisfied, but there are quasimodular lattices which are not modular. It is important to note that the lattice L of subspaces of H is not modular¹. On the other hand it can be shown that the relation (2.4) is not always true in an orthocomplemented lattice. Therefore this relation is an additional nontrivial property in the orthocomplemented lattice L_0 .

In summarizing these results we find that the lattice of subspaces of the Hilbert space is a σ -complete, atomic, orthocomplemented lattice L_0 which is quasimodular in the sense of (2.4). A lattice which has just these properties will be denoted by L_q .

It is the goal of this paper to investigate the question whether the lattice L_q of the subspaces of the Hilbert space can be interpreted as a propositional calculus. It is easy to see that the subspaces themselves can be interpreted as propositions and that the operations $a \wedge b$, $a \vee b$ and a' can be interpreted as “ a and b ”, “ a or b ” and “not a ”, respectively. The relation $a \leq b$ between two elements of L_q has the meaning of “ a implies b ”. However according to the arguments discussed in §1 this correspondence between lattice operations and logical operations is not sufficient if one wants to consider L_q as a propositional calculus. In order to formulate the well known logical theorems concerning iterated implications in terms of lattice theory it is necessary to show that for any two elements a and b there exists an element $q(a, b)$ in L_q which has the essential properties of the implication. It is, however, quite impossible to introduce in L_q the “implication” in the same way as in the lattice L_B of classical logic. The axioms (1.1) and (1.2) are not valid in L_q , since from (1.1) and (1.2) it follows that the lattice considered is distributive.

Fortunately it is possible to find two somewhat weaker axioms which are still valid in L_q and which allow for a definition of an operation which has many properties in common with the classical implication. The existence of this operation, which we will call “quasi-implication”, makes it possible to interpret L_q as a propositional calculus. Since we are dealing here from the very beginning with an orthocomplemented lattice L_0 with θ and $\mathbf{1}$ elements, the relation (2.3) which corresponds to the “tertium non datur” in the propositional calculus is always true. Therefore the quasi-implication which we will define in the next paragraph corresponds to the “classical implication” which satisfies Peirce’s law (1.3). In fact it will be shown in theorem V that also the quasi-implication defined here satisfies this and the relation (1.4), which in L_I are equivalent.

§ 3. Definition of the Quasi Implication in L_q

We start from a σ -complete atomic orthocomplemented lattice L_0 . In addition to the axioms of L_0 we postulate two further axioms (3.1) and (3.2), which will be somewhat weaker than (1.1) and (1.2) and still valid in L_q . Moreover it will be shown that the axioms are necessary and sufficient for the quasimodularity of the lattice considered (theorem I and II) and that they determine the quasi-implication uniquely (theorem III). Furthermore a few important relations, which are known to be valid for the classical implication, will be shown to be satisfied also by the quasi-implication (theorem IV, V, VI).

If for any pair of elements $a, b \in L_0$ there exists an element $q(a, b)$ which satisfies the two relations

$$a \wedge q(a, b) \leq b, \quad (3.1)$$

$$a \wedge x \leq b \Rightarrow a' \vee (a \wedge x) \leq q(a, b) \quad (3.2)$$

the lattice will be called *quasi-implicative*. The element $q(a, b)$ will be called *quasi-implication* in contrast to the implication defined by (1.1) and (1.2).

Theorem I

A lattice L_0 which is quasi-implicative is quasimodular in the sense of (2.4) i.e. for three arbitrary elements a, x and y the relation holds

$$x \leq a, y \leq a' \Rightarrow a \wedge (x \vee y) \leq (a \wedge x) \vee (a \wedge y) \quad (3.3)$$

Proof: If we put $b = (a \wedge x) \vee (a \wedge y)$ we get

$$a \wedge x \leq b,$$

$$a \wedge y \leq b,$$

and from (3.2)

$$\begin{aligned} a' \vee (a \wedge x) &\leq q(a, b), \\ a' \vee (a \wedge y) &\leq q(a, b), \end{aligned}$$

and $a' \vee (a \wedge x) \vee (a \wedge y) \leq q(a, b)$.

Supposing that the relations on the left hand side of (3.3) are valid we get

$$\begin{aligned} x = a \wedge x &\leq a' \vee (a \wedge x) \leq q(a, b), \\ y \leq a' &\leq a' \vee (a \wedge y) \leq q(a, b) \end{aligned}$$

and therefore

$$x \vee y \leq q(a, b).$$

Using the axiom (3.1) and substituting b in this relation we obtain

$$a \wedge (x \vee y) \leq a \wedge q(a, b) \leq (a \wedge x) \vee (a \wedge y)$$

and $a \wedge (x \vee y) \leq (a \wedge x) \vee (a \wedge y)$

which finishes the proof.

Theorem II

In a lattice L_q for any two elements a and b there exists an element $q(a, b)$ which satisfies (3.1) and (3.2). Therefore L_q is a quasi-implicative lattice. The quasi-implication $q(a, b)$ is given by

$$q(a, b) = a' \vee (a \wedge b). \quad (3.4)$$

Proof:

1) Axiom (3.1). If we put in (2.4) $\alpha = a \wedge b$, $\beta = a$ and $\gamma = a'$ we get

$$a \wedge (a' \vee (a \wedge b)) = a \wedge b$$

from which on account of $q(a, b) = a' \vee (a \wedge b)$ we obtain

$$a \wedge q(a, b) = a \wedge b \leq b.$$

2) Axiom (3.2). From $a \wedge x \leq b$ we get $a \wedge x \leq a \wedge b$ and therefore

$$a' \vee (a \wedge x) \leq a' \vee (a \wedge b) = q(a, b).$$

Here the quasimodularity (2.4) is not necessary.

Theorem III

In a lattice L_q there exists only one element $q(a, b)$ which satisfies the relations (3.1) and (3.2).

Proof: The existence of one element $q(a, b)$ which satisfies (3.1) and (3.2) has been proved in theorem II. This element is given by $q(a, b) = a' \vee (a \wedge b)$. Let us assume that there is still another element $y(a, b)$ which also satisfies the relations

$$a \wedge y(a, b) \leq b, \quad (3.5)$$

$$a \wedge x \leq b \Rightarrow a' \vee (a \wedge x) \leq y(a, b). \quad (3.6)$$

If we put $x = b$ we obtain from (3.6)

$$a' \vee (a \wedge b) \leq y(a, b)$$

and therefore

$$q(a, b) \leq y(a, b). \quad (3.7)$$

Furthermore from (3.5) we obtain

$$a \wedge y(a, b) \leq a \wedge b \leq a' \vee (a \wedge b)$$

and therefore

$$a' \vee (a \wedge y) \leq q(a, b). \quad (3.8)$$

If we put in (2.4) $\alpha = a' \vee (a \wedge y)$, $\beta = a' \vee y$ and $\gamma = \beta'$ the left hand side of (2.4) is always satisfied and we get

$$(a' \vee y) \wedge (a' \vee (a \wedge y) \vee (a' \vee y)') \leq a' \vee (a \wedge y) \quad (3.9)$$

and on account of (3.8)

$$(a' \vee y) \wedge (a' \vee (a \wedge y) \vee (a' \vee y)') \leq q(a, b). \quad (3.10)$$

If we put in (3.6) $x = a'$ the left hand side of (3.6) is always true and we get $a' \leq y$. Therefore we obtain together with (2.2)

$$a' \vee y = y, \quad (a' \vee y)' = y'$$

and

$$y \wedge (a' \vee y' \vee (a \wedge y)) \leq q(a, b) \quad (3.11)$$

from which we finally get

$$y \wedge ((a \wedge y)' \vee (a \wedge y)) = y \leq q(a, b). \quad (3.12)$$

From (3.7) and (3.12) it results that $y(a, b) = q(a, b)$ which finishes the proof.

Theorem IV

If in a lattice L_0 there exists an element $q(a, b)$ which satisfies (3.1) and (3.2) it satisfies also the relation

$$a \leq b \Leftrightarrow \mathbf{I} \leq q(a, b). \quad (3.13)$$

Remark: In logical terms (3.13) means, that the proposition $q(a, b)$ is true if and only if the relation $a \leq b$ holds.

Proof:

1) If $a \leq b$ we have $a \wedge \mathbf{I} \leq b$ and from (3.2) we get

$$a' \vee (a \wedge \mathbf{I}) = a' \vee a = \mathbf{I} \leq q(a, b).$$

2) If $\mathbf{I} \leq q(a, b)$ we have $a \leq q(a, b)$ and from (3.1) we get

$$a \wedge q(a, b) = a \leq b.$$

Corollary IV.1. If we put $b = a$ we get

$$q(a, a) = \mathbf{I}. \quad (3.14)$$

Corollary IV.2. If we put $a = \mathbf{0}$ and $b = \mathbf{I}$ we get

$$q(\mathbf{0}, a) = q(a, \mathbf{I}) = \mathbf{I}. \quad (3.15)$$

Corollary IV.3. Since the only element in L_0 which satisfies (3.1) and (3.2) is $a' \vee (a \wedge b)$ we get from (3.13)

$$a \leq b \Leftrightarrow \mathbf{I} \leq a' \vee (a \wedge b).$$

Theorem V

The quasi-implication $q(a, b) = a' \vee (a \wedge b)$ satisfies the relations

$$q(q(a, b), a) \leq a, \quad (3.16)$$

$$a \vee q(a, b) = \mathbf{I}. \quad (3.17)$$

Remark: In an implicative lattice L_I these relations are equivalent. If they are valid, if a zero element exists and if the complement is defined by $a' = c(a, \mathbf{0})$, L_I is a Boolean lattice and the *tertium non datur* $a \vee a' = \mathbf{I}$ is true.

Proof:

1) (3.16): From $q(a, b) = a' \vee (a \wedge b)$ we get

$$q(q(a, b), a) = q'(a, b) \vee (q(a, b) \wedge a)$$

and therefore

$$q(q(a, b), a) \leq a.$$

2) (3.17): From $q(a, b) = a' \vee (a \wedge b)$ we get

$$a \vee q(a, b) = (a \vee a') \vee (a \wedge b) = a \vee a' = \mathbf{I}.$$

Theorem VI

The relation $q(a, b) = \mathbf{I}$ is transitive, i.e. $\mathbf{I} = q(a, b)$ and $\mathbf{I} = q(b, x)$ implies $\mathbf{I} = q(a, x)$.

Remark: The transitivity of the relation $\mathbf{I} = c(a, b)$ concerning the classical implication implies that the lattice is Boolean (cf. Ref. ¹¹, p. 174, and the literature quoted there). The transitivity of the relation $\mathbf{I} = q(a, b)$, which uses the quasi-implication, has not this consequence.

Proof: If we assume $\mathbf{I} = q(a, b)$ and $\mathbf{I} = q(b, x)$ from theorem IV we get, that $a \leq b$ and $b \leq x$ holds. Therefore from the transitivity of the relation \leq it follows $a \leq x$. Using again theorem IV we obtain $\mathbf{I} = q(a, x)$.

Concluding Remarks

It has been shown in the preceding paragraph, that the quasi-implication $q(a, b)$ defined here satisfies many important relations in L_q which are known to be valid for the classical implication in L_B . We have confined ourselves here to those properties which are described by the theorems I–VI and which seem to us to be the most interesting ones. There are many other relations known from the classical implication which are also true for the quasi-implication. Therefore it seems to us that the quasi-implication might be considered as a convenient generalisation of the classical implication, which makes it possible to interpret the lattice L_q as a propositional calculus sometimes called “quantum logic”. However it should be mentioned that there is an important relation which holds for the classical implication in L_I but which is not satisfied by the quasi-implication in L_q . In an implicative lattice L_I it follows from axiom (1.2), that

$$a \leq c(b, a)$$

which is not generally true in L_q . In a classical implicative lattice with zero element it follows from this relation that

$$a = (a \wedge b) \vee (a \wedge b')$$

which means that the propositions a and b are compatible, i.e. the physical properties corresponding to a and b are simultaneously measurable. It is well known that this is in general not the case. (For a more detailed discussion of this point cf. Ref. ², p. 80 and Ref. ¹², §2). However it seems to us that these restrictions, which come from the incompatibility of quantum mechanical propositions, do not invalidate the interpretations of the lattice L_q as a propositional calculus.

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